

Another example of duality between game-theoretic and measure-theoretic probability*

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August 10, 2016

Abstract

This paper makes a small step towards a non-stochastic version of superhedging duality relations in the case of one traded security with a continuous price path. Namely, we prove the coincidence of game-theoretic and measure-theoretic expectation for lower semicontinuous positive functionals. We consider a new broad definition of game-theoretic probability, leaving the older narrower definitions for future work.

1 Introduction

The words like “positive” and “increasing” will be understood in the wide sense (e.g., a is positive if $a \geq 0$), and the qualifier “strictly” will indicate the narrow sense (e.g., a is strictly positive if $a > 0$). The set of all continuous real-valued functions on a topological space X is denoted, as usual, $C(X)$, and its subset consisting of positive functions is denoted $C^+(X)$. We abbreviate expressions such as $C([0, T])$ and $C^+([0, T])$, where $T > 0$, $C([0, \infty))$, and $C^+([0, \infty))$ to $C[0, T]$, $C^+[0, T]$, $C[0, \infty)$, and $C^+[0, \infty)$, respectively, and let $C_a[0, T]$, $C_a^+[0, T]$, $C_a[0, \infty)$, and $C_a^+[0, \infty)$ stand for the subsets of these sets consisting of the functions f satisfying $f(0) = a$, for a given constant a .

Let $\mathbb{N} := \{1, 2, \dots\}$ be the set of all strictly positive integers, and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ be the set of all positive integers.

As usual $a \wedge b$ stands for minimum of a and b and $a \vee b$ for their maximum. In this paper, the operators \wedge and \vee have higher precedence than the arithmetic operators: e.g., $a + b \wedge c$ means $a + (b \wedge c)$. Other conventions of this kind are that:

- Cartesian product \times has higher precedence than union \cup ; so that, e.g., $A \cup \{1\} \times [0, \infty)$ means $A \cup (\{1\} \times [0, \infty))$;
- implicit multiplication (not using a multiplication sign such as \times or \cdot) has higher precedence than division; so that, e.g., S/NL means $S/(NL)$.

*The version of this paper at <http://probabilityandfinance.com> (Working Paper 46) is updated more often.

In our informal discussions we will use symbols \approx for approximate equality and \lesssim and \gtrsim for approximate inequalities.

In this paper we consider a finite time interval $[0, T]$ where $T \in (0, \infty)$; without loss of generality we set $T := 1$.

2 The main result

The *sample space* used in this paper, $\Omega := C_1^+[0, 1]$, is the set of all positive continuous functions $\omega : [0, 1] \rightarrow [0, \infty)$ such that $\omega(0) = 1$. Intuitively, the functions in Ω are price paths of a financial security whose initial price serves as the unit for measuring its later prices.

We equip Ω with the usual σ -algebra \mathcal{F} , i.e., the smallest σ -algebra making all functions $\omega \in \Omega \mapsto \omega(t)$, $t \in [0, 1]$, measurable. A *process* (more fully, an *adapted process*) \mathfrak{S} is a family of extended random variables $\mathfrak{S}_t : \Omega \rightarrow [-\infty, \infty]$, $t \in [0, \infty)$, such that, for all $\omega, \omega' \in \Omega$ and all $t \in [0, \infty)$,

$$\omega|_{[0, t]} = \omega'|_{[0, t]} \implies \mathfrak{S}_t(\omega) = \mathfrak{S}_t(\omega');$$

its *sample paths* are the functions $t \in [0, 1] \mapsto \mathfrak{S}_t(\omega)$. A *stopping time* is an extended random variable $\tau : \Omega \rightarrow [0, \infty]$ such that, for all $\omega, \omega' \in \Omega$,

$$\omega|_{[0, \tau(\omega) \wedge 1]} = \omega'|_{[0, \tau(\omega) \wedge 1]} \implies \tau(\omega) = \tau(\omega'),$$

where $\omega|_A$ stands for the restriction of ω to $A \subseteq [0, 1]$. For any stopping time τ , the σ -algebra \mathcal{F}_τ is defined as the family of all events $E \in \mathcal{F}$ such that, for all $\omega, \omega' \in \Omega$,

$$(\omega|_{[0, \tau(\omega) \wedge 1]} = \omega'|_{[0, \tau(\omega) \wedge 1]}, \omega \in E) \implies \omega' \in E. \quad (1)$$

Therefore, a random variable X is \mathcal{F}_τ -measurable if and only if, for all $\omega, \omega' \in \Omega$,

$$\omega|_{[0, \tau(\omega) \wedge 1]} = \omega'|_{[0, \tau(\omega) \wedge 1]} \implies X(\omega) = X(\omega').$$

Remark 1. Our definitions (convenient for the purposes of this paper) are equivalent to the standard ones by Galmarino's test ([3], IV.100).

First we define game-theoretic probability and expectation, partly following Perkowski and Prömel [8, 7, 1] (this is a “broad” definition making our task easier; the older “narrow” definition of [11] is much more conservative and might require stronger assumptions for our main result to hold true; another broad definition was given in [12]). A *simple trading strategy* G consists of an increasing sequence of stopping times $\tau_1 \leq \tau_2 \leq \dots$ (we may assume, without loss of generality, $\tau_n \in [0, 1] \cup \{\infty\}$ and $\tau_n < \tau_{n+1}$ unless $\tau_n = \infty$) and, for each $n = 1, 2, \dots$, a bounded \mathcal{F}_{τ_n} -measurable function h_n . It is required that, for each $\omega \in \Omega$, $\lim_{n \rightarrow \infty} \tau_n(\omega) = \infty$. To such G and an *initial capital* $c \in \mathbb{R}$ corresponds the *simple capital process*

$$\mathcal{K}_t^{G, c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega) (\omega(\tau_{n+1}(\omega) \wedge t) - \omega(\tau_n(\omega) \wedge t)), \quad t \in [0, \infty); \quad (2)$$

the value $h_n(\omega)$ will be called the *bet* (or *bet on* ω , or *stake*) at time τ_n , and $\mathcal{K}_t^{G,c}(\omega)$ will be called the *capital* at time t . For $c \geq 0$, let \mathcal{C}_c be the class of positive functionals of the form $\mathcal{K}_1^{G,c}$, G ranging over simple trading strategies; intuitively, these are the functionals that can be hedged with initial capital c by a simple strategy that does not risk bankruptcy (notice that $\forall \omega : \mathcal{K}_1^{G,c}(\omega) \geq 0$ implies $\forall \omega \forall t : \mathcal{K}_t^{G,c}(\omega) \geq 0$).

A class \mathcal{C} of functionals $F : \Omega \rightarrow [0, \infty]$ is *lim inf-closed* if $F \in \mathcal{C}$ whenever there is a sequence F_1, F_2, \dots of functionals in \mathcal{C} such that

$$\forall \omega \in \Omega : F(\omega) \leq \liminf_{n \rightarrow \infty} F_n(\omega). \quad (3)$$

The intuition is that if F_1, F_2, \dots can be superhedged, so can F in the limit. It is clear that for each class \mathcal{C} of functionals there is a smallest lim inf-closed class, denoted $\overline{\mathcal{C}}$, containing \mathcal{C} .

The *upper game-theoretic expectation* of a functional $F : \Omega \rightarrow [0, \infty]$ is defined to be

$$\overline{\mathbb{E}}^g(F) := \inf \{c \mid F \in \overline{\mathcal{C}_c}\}. \quad (4)$$

where \mathcal{C}_c is as defined above. The *upper game-theoretic probability* of $E \subseteq \Omega$ is $\overline{\mathbb{P}}^g(E) := \overline{\mathbb{E}}^g(\mathbf{1}_E)$, $\mathbf{1}_E$ being the indicator function of E .

The *upper measure-theoretic expectation* of F is defined to be

$$\overline{\mathbb{E}}^m(F) := \sup_P \int F dP,$$

where P ranges over all *martingale measures*, i.e., probability measures on Ω under which the process $X_t(\omega) := \omega(t)$ is a martingale, and \int stands for upper integral. The *upper measure-theoretic probability* of $E \subseteq \Omega$ is $\overline{\mathbb{P}}^m(E) := \overline{\mathbb{E}}^m(\mathbf{1}_E)$.

Now we can state our main result, Theorem 2, in which “lower semicontinuous” refers to the standard topology on Ω generated by the usual uniform metric

$$\rho_U(\omega, \omega') := \sup_{t \in [0,1]} |\omega(t) - \omega'(t)|. \quad (5)$$

Theorem 2. *For any lower semicontinuous functional $F : \Omega \rightarrow [0, \infty]$,*

$$\overline{\mathbb{E}}^g(F) = \overline{\mathbb{E}}^m(F)$$

(the inequality \geq holding for all $F : \Omega \rightarrow [0, \infty]$).

An earlier result of the same kind is the discrete-time Theorem 1 of [10].

3 Proof of Theorem 2

In this section we prove the coincidence of $\overline{\mathbb{E}}^g$ and $\overline{\mathbb{E}}^m$ on “simple” (lower semicontinuous in this version of the paper) positive functionals. We prove the inequality \geq in Subsection 3.1 and the inequality \leq in Subsection 3.2. Notice that we can ignore $\omega \in \Omega$ such that $0 = \omega(t) < \omega(s)$ for some $0 \leq t < s$.

On a few occasions we will use the following simple lemma.

Lemma 3. *The functions $\overline{\mathbb{E}}^g$ and $\overline{\mathbb{E}}^m$ are σ -subadditive: for any sequence of positive functionals F_1, F_2, \dots (taking values in $[0, \infty]$),*

$$\overline{\mathbb{E}}^g \left(\sum_{n=1}^{\infty} F_n \right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{E}}^g(F_n), \quad (6)$$

$$\overline{\mathbb{E}}^m \left(\sum_{n=1}^{\infty} F_n \right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{E}}^m(F_n). \quad (7)$$

(And therefore, the set functions $\overline{\mathbb{P}}^g$ and $\overline{\mathbb{P}}^m$ are outer measures.)

Proof. We can deduce (7) from the σ -subadditivity of $F \mapsto \int F dP$: indeed, for each $\epsilon > 0$,

$$\begin{aligned} \overline{\mathbb{E}}^m \left(\sum_{n=1}^{\infty} F_n \right) &= \sup_P \int \sum_{n=1}^{\infty} F_n dP \leq \int \sum_{n=1}^{\infty} F_n dP_0 + \epsilon \leq \sum_{n=1}^{\infty} \int F_n dP_0 + \epsilon \\ &\leq \sum_{n=1}^{\infty} \sup_P \int F_n dP + \epsilon = \sum_{n=1}^{\infty} \overline{\mathbb{E}}^m(F_n) + \epsilon, \end{aligned}$$

where P_0 is a martingale measure.

As for (6), we start from a new definition of $\overline{\mathcal{C}}_c$. Define \mathcal{C}_c^α by transfinite induction over the countable ordinals α (see, e.g., [3], 0.8) as follows:

- $\mathcal{C}_c^0 := \mathcal{C}_c$;
- for $\alpha > 0$, $F \in \mathcal{C}_c^\alpha$ if and only if there exists a sequence F_1, F_2, \dots of functionals in $\mathcal{C}_c^{<\alpha} := \cup_{\beta < \alpha} \mathcal{C}_c^\beta$ such that (3) holds.

It is easy to check that $\overline{\mathcal{C}}_c$ is the union of the nested family \mathcal{C}_c^α over all countable ordinals α .

First we prove finite subadditivity ((6) with ∞ replaced by a natural number), which will immediately follow from

$$(F_i \in \overline{\mathcal{C}}_{c_i}, i = 1, \dots, n) \implies \left(\sum_{i=1}^n F_i \in \overline{\mathcal{C}}_{\sum_{i=1}^n c_i} \right).$$

It suffices to prove, for each countable ordinal α ,

$$(F_i \in \mathcal{C}_{c_i}^\alpha, i = 1, \dots, n) \implies \left(\sum_{i=1}^n F_i \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha \right) \quad (8)$$

(this is the implication that we will actually need below). This is true for $\alpha = 0$ (by the definition of a simple trading strategy), so we fix a countable ordinal $\alpha > 0$ and assume that the statement holds for all ordinals below α . Let us also assume the antecedent of (8). For each $i \in \{1, \dots, n\}$ let $F_i^j \in \mathcal{C}_c^{<\alpha}$, $j = 1, 2, \dots$, be a sequence such that

$$\forall \omega \in \Omega : F_i(\omega) \leq \liminf_{j \rightarrow \infty} F_i^j(\omega).$$

For each j , the inductive assumption gives

$$\sum_{i=1}^n F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^{\leq \alpha}$$

(since there are finitely many i , there is $\beta = \beta_j < \alpha$ such that $F_i^j \in \mathcal{C}_{c_i}^\beta$ for all $i \in \{1, \dots, n\}$). By the definition of \mathcal{C}^α ,

$$\liminf_{j \rightarrow \infty} \sum_{i=1}^n F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha,$$

which implies, by the Fatou lemma,

$$\sum_{i=1}^n \liminf_{j \rightarrow \infty} F_i^j \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha,$$

which in turn implies

$$\sum_{i=1}^n F_i \in \mathcal{C}_{\sum_{i=1}^n c_i}^\alpha.$$

The countable subadditivity (6) now follows immediately from Lemma 5 below:

$$\begin{aligned} \overline{\mathbb{E}}^g \left(\sum_{n=1}^{\infty} F_n \right) &= \overline{\mathbb{E}}^g \left(\liminf_{N \rightarrow \infty} \sum_{n=1}^N F_n \right) \leq \liminf_{N \rightarrow \infty} \overline{\mathbb{E}}^g \left(\sum_{n=1}^N F_n \right) \\ &\leq \liminf_{N \rightarrow \infty} \sum_{n=1}^N \overline{\mathbb{E}}^g(F_n) = \sum_{n=1}^{\infty} \overline{\mathbb{E}}^g(F_n). \quad \square \end{aligned}$$

Remark 4. The original “broad” definition of game-theoretic probability and expectation in [8] is given by (4) with \mathcal{C}_c^1 in place of $\overline{\mathcal{C}}_c$.

The following lemma (already used in the proof of Lemma 3 above) is the analogue of the Fatou lemma for the broad definition of game-theoretic probability.

Lemma 5. *For any sequence of positive functionals F_1, F_2, \dots ,*

$$\overline{\mathbb{E}}^g \left(\liminf_{n \rightarrow \infty} F_n \right) \leq \liminf_{n \rightarrow \infty} \overline{\mathbb{E}}^g(F_n). \quad (9)$$

Proof. Let c be the right-hand side of (9) and $\epsilon > 0$. There is a strictly increasing sequence $n_1 < n_2 < \dots$ such that $\overline{\mathbb{E}}^g(F_{n_i}) < c + \epsilon$ for all i . Since $F_{n_i} \in \overline{\mathcal{C}}_{c+\epsilon}$ for all i , we have $\liminf_{i \rightarrow \infty} F_{n_i} \in \overline{\mathcal{C}}_{c+\epsilon}$, which implies $\liminf_{n \rightarrow \infty} F_n \in \overline{\mathcal{C}}_{c+\epsilon}$, which in turn implies $\overline{\mathbb{E}}^g(\liminf_{n \rightarrow \infty} F_n) \leq c + \epsilon$. Since ϵ can be made arbitrarily small, this completes the proof. \square

3.1 Inequality \geq

The goal of this subsection is to prove

$$\overline{\mathbb{E}}^m(F) \leq \overline{\mathbb{E}}^g(F) \quad (10)$$

for all functionals $F : \Omega \rightarrow [0, \infty]$ (we will not need the assumptions that F is bounded or measurable).

First we will prove

$$\mathbb{E}_P \left(\mathcal{K}_1^{G,c} - c \right) \leq 0 \quad (11)$$

for all martingale measures P , where G is a simple trading strategy whose stopping times and bets will be denoted τ_1, τ_2, \dots and h_1, h_2, \dots , respectively, and c is an initial capital. Fix such a P . By the Fatou lemma (applied to the partial sums in (2)), it suffices to prove (11) assuming that the sequence of stopping time is finite: $\tau_n = \infty$ for all $n > N$ for a given $N \in \mathbb{N}$ (which in turn implies that the bets h_n are bounded in absolute value by a given constant).

For each $k = 1, 2, \dots$, set $\tau_n^k := 2^{-k} \lceil 2^k \tau_n \rceil$ and let \mathfrak{S}^k be the simple capital process corresponding to initial capital $\mathfrak{S}_0^k = c$, stopping times τ_n^k , and bets h_n (remember that our definition of a simple trading strategy allows $\tau_n = \tau_{n+1}$). It is easy to check that, for all k and $n = 0, \dots, 2^k - 1$,

$$\mathbb{E}_P \left(\mathfrak{S}_{(n+1)2^{-k}}^k - \mathfrak{S}_{n2^{-k}}^k \right) = 0; \quad (12)$$

indeed, the difference $\mathfrak{S}_{(n+1)2^{-k}}^k - \mathfrak{S}_{n2^{-k}}^k$ is the product of the bounded $\mathcal{F}_{n2^{-k}}$ -measurable function

$$h := \sum_{i=1}^N h_i \mathbf{1}_{\{\tau_i^k = n2^{-k}, \tau_{i+1}^k > n2^{-k}\}}$$

and the martingale difference $\omega((n+1)2^{-k}) - \omega(n2^{-k})$, and so

$$\begin{aligned} & \mathbb{E}_P \left(\mathfrak{S}_{(n+1)2^{-k}}^k - \mathfrak{S}_{n2^{-k}}^k \right) \\ &= \mathbb{E}_{P(d\omega)} \left(\mathbb{E}_{P(d\omega)} \left(h(\omega) (\omega((n+1)2^{-k}) - \omega(n2^{-k})) \mid \mathcal{F}_{n2^{-k}} \right) \right) \\ &= \mathbb{E}_{P(d\omega)} \left(h(\omega) \mathbb{E}_{P(d\omega)} \left((\omega((n+1)2^{-k}) - \omega(n2^{-k})) \mid \mathcal{F}_{n2^{-k}} \right) \right) = 0. \end{aligned}$$

Summing (12) over n (of which there are finitely many),

$$\mathbb{E}_P \left(\mathfrak{S}_1^k - c \right) = 0,$$

which in turn implies, by the Fatou lemma, (11).

We will complete the proof of (10) by transfinite induction, as in Lemma 3. Rewrite (10) as $\overline{\mathbb{E}}^m(F) \leq c$ for all $F \in \overline{\mathcal{C}}_c$. Fix c and $F \in \overline{\mathcal{C}}_c$. In the previous paragraph we checked that $\overline{\mathbb{E}}^m(F) \leq c$ if $F \in \mathcal{C}_c^0$. Therefore, it remains to prove, for a given countable ordinal $\alpha > 0$, that $\overline{\mathbb{E}}^m(F) \leq c$ assuming that $F \in \mathcal{C}_c^\alpha$ and that $\overline{\mathbb{E}}^m(G) \leq c$ for all $G \in \mathcal{C}_c^{<\alpha}$. Let $F_n \in \mathcal{C}_c^{<\alpha}$, $n = 1, 2, \dots$, be

a sequence of functionals such that $F \leq \liminf_n F_n$. Suppose $\overline{\mathbb{E}}^m(F) > c$ and find a martingale measure P such that $c < \int F dP$. We get a contradiction by the Fatou lemma and the inductive assumption:

$$c < \int F dP \leq \int \liminf_{n \rightarrow \infty} F_n dP \leq \liminf_{n \rightarrow \infty} \int F_n dP \leq \liminf_{n \rightarrow \infty} c = c.$$

3.2 Inequality \leq

In this section we will prove that

$$\overline{\mathbb{E}}^g(F) \leq \overline{\mathbb{E}}^m(F). \quad (13)$$

Since $\overline{\mathbb{E}}^g(F)$ is defined as an infimum and $\overline{\mathbb{E}}^m(F)$ as a supremum, it suffices to construct a martingale measure P and a superhedging capital process for a given lower semicontinuous positive functional F such that $\int F dP$ is close to (or greater than) the initial capital of the process.

3.2.1 Reductions I

The goal of this section is to show that, without loss of generality, we can assume that the functional F is bounded and lower semicontinuous in a stronger sense.

For a general lower semicontinuous $F : \Omega \rightarrow [0, \infty]$ and $n \in \mathbb{N}$, set $F_n(\omega) := F(\omega) \wedge n$. Assuming $\overline{\mathbb{E}}^g(F_n) \leq \overline{\mathbb{E}}^m(F_n)$ for all n , let us prove $\overline{\mathbb{E}}^g(F) \leq \overline{\mathbb{E}}^m(F)$. Set $c := \overline{\mathbb{E}}^m(F) + \epsilon$ for a small $\epsilon > 0$. Since

$$\overline{\mathbb{E}}^g(F_n) \leq \overline{\mathbb{E}}^m(F_n) \leq \overline{\mathbb{E}}^m(F) < c,$$

we have $F_n \in \overline{\mathcal{C}}_c$. Since $\overline{\mathcal{C}}_c$ is \liminf -closed, we have

$$F = \liminf_{n \rightarrow \infty} F_n \in \overline{\mathcal{C}}_c$$

and, therefore, $\overline{\mathbb{E}}^g(F) \leq c$. Since ϵ can be arbitrarily small, this completes the proof of $\overline{\mathbb{E}}^g(F) \leq \overline{\mathbb{E}}^m(F)$. Therefore, we can, and will, assume that F is bounded above.

In the rest of this paper, instead of the uniform metric (5) we will consider the Hausdorff metric

$$\begin{aligned} \rho_H(\omega, \omega') := H(\bar{\omega}, \bar{\omega}') := & \sup_{(t, x) \in \bar{\omega}} \inf_{(t', x') \in \bar{\omega}'} \|(t - t', x - x')\| \\ & \vee \sup_{(t', x') \in \bar{\omega}'} \inf_{(t, x) \in \bar{\omega}} \|(t - t', x - x')\|, \end{aligned} \quad (14)$$

where $\|\cdot\| = \|\cdot\|_\infty$ stands for the ℓ_∞ norm $\|(a, b)\| := |a| \vee |b|$ in \mathbb{R}^2 and each element ω of Ω is mapped to the set $\bar{\omega} \subseteq [0, 1] \times [0, \infty)$ defined to be the union $\text{graph}(\omega) \cup \{1\} \times [0, \infty)$ of the graph of ω and the ray $\{1\} \times [0, \infty)$.

Remark 6. Notice that the metrics ρ_U and ρ_H lead to different topologies: e.g., there is an unbounded sequence ω_n of elements of Ω such that $\omega_n \rightarrow 0$ in ρ_H . The ℓ_∞ norm (used in our definition of ρ_H) is, of course, equivalent to the Euclidean norm ℓ_2 , but sometimes it leads to slightly simpler formulas. An example of a functional $F : \Omega \rightarrow \mathbb{R}$ continuous in ρ_H is $F(\omega) := F'(\omega|_{[0,1-\epsilon]})$, where F' is a functional on $C_1^+[0, 1 - \epsilon]$ continuous in the uniform metric and $\epsilon \in (0, 1)$ is a strictly positive constant.

Remark 7. On the other hand, the topologies generated by the metrics ρ_U and ρ_H lead to the same Borel σ -algebra. Since the topology generated by ρ_U is finer than the one generated by ρ_H , it suffices to check that every ρ_U -Borel set is a ρ_H -Borel set. Since the ρ_U -topology is separable, it suffices to check that every open ball in ρ_U is a ρ_H -Borel set. This is easy; moreover, every open ball in ρ_U is the intersection of a sequence of ρ_H -open sets.

Let us check that in Theorem 2 we can further assume that F is lower semicontinuous in the Hausdorff topology on Ω (this observation develops the end of Remark 6). Suppose that Theorem 2 holds for all (bounded) positive functionals that are lower semicontinuous in the Hausdorff topology. It is clear that we can replace the sample space Ω by the sample space $\Omega^* := C_1^+[0, 2]$; let us do so. Now let F be a lower semicontinuous (in the usual uniform topology) positive functional on Ω . Define

$$F^*(\omega) := F(\omega|_{[0,1]}), \quad \omega \in \Omega^*.$$

Then F^* is lower semicontinuous in the Hausdorff metric on Ω^* (defined by (14) where the ray $\{1\} \times [0, \infty)$ in the definition of $\bar{\omega}$ is replaced by $\{2\} \times [0, \infty)$). Indeed, for any constant c , the set $\{F^* > c\}$ is open: if $\omega_n \rightarrow \omega$ in the Hausdorff metric on $C_1^+[0, 2]$, then $\omega_n|_{[0,1]} \rightarrow \omega|_{[0,1]}$ in the usual topology (had $\omega_n|_{[0,1]}$ not converged to $\omega|_{[0,1]}$ in the usual topology, we could have found $\epsilon > 0$ and $t_n \in [0, 1]$ such that $|\omega_n(t_n) - \omega(t_n)| > \epsilon$ for infinitely many n and arrived at a contradiction by considering a limit point of those t_n), and so

$$\forall n : F^*(\omega_n) \leq c \iff \forall n : F(\omega_n|_{[0,1]}) \leq c \implies F(\omega|_{[0,1]}) \leq c \iff F^*(\omega) \leq c.$$

Therefore, our assumption (the non-trivial part of Theorem 2 for the Hausdorff metric) gives

$$\overline{\mathbb{E}}^g(F^*) \leq \overline{\mathbb{E}}^m(F^*),$$

and it suffices to prove $\overline{\mathbb{E}}^g(F) \leq \overline{\mathbb{E}}^g(F^*)$ and $\overline{\mathbb{E}}^m(F^*) \leq \overline{\mathbb{E}}^m(F)$.

First let us check that $\overline{\mathbb{E}}^g(F) \leq \overline{\mathbb{E}}^g(F^*)$. This follows from the class $\overline{\mathcal{C}}_c$ dominating the class $\overline{\mathcal{C}}_c^*$ for all $c > 0$, where the class \mathcal{C}_c is as defined above, the class \mathcal{C}_c^* is the analogue of this class for the time interval $[0, 2]$ rather than $[0, 1]$, and a class \mathcal{A} of functionals on Ω is said to *dominate* a class \mathcal{B} of functionals on Ω^* if for any $G \in \mathcal{B}$ there exists $G' \in \mathcal{A}$ that *dominates* G in the sense that, for any $\omega \in \Omega$,

$$G'(\omega) \geq \mathbb{E}_{W(d\xi)}(G(\omega\xi))$$

where W is the Wiener measure on $C_0[0, 1]$ and $\omega\xi : [0, 2] \rightarrow [0, \infty)$ is the continuous combination of ω and ξ defined as follows:

$$(\omega\xi)(t) := \begin{cases} \omega(t) & \text{if } t \in [0, 1] \\ \omega(1) + \xi(t-1) & \text{if } t \in [1, \tau] \\ 0 & \text{if } t \in (\tau, 1] \end{cases}$$

where

$$\tau := \inf\{t \in [1, 2] \mid \omega(1) + \xi(t-1) = 0\}$$

with $\inf \emptyset := 2$. Indeed, assuming that $\overline{\mathcal{C}_c}$ dominates $\overline{\mathcal{C}_c^*}$ for all $c > 0$, we obtain

$$\overline{\mathbb{E}^g}(F) = \inf\{c \mid F \in \overline{\mathcal{C}_c}\} \leq \inf\{c \mid F^* \in \overline{\mathcal{C}_c^*}\} = \overline{\mathbb{E}^g}(F^*),$$

where the inequality follows from the fact that, whenever $F^* \in \overline{\mathcal{C}_c^*}$, F^* is dominated by some $G \in \overline{\mathcal{C}_c}$, which implies

$$\forall \omega \in \Omega : F(\omega) = \mathbb{E}_{W(d\xi)}(F^*(\omega\xi)) \leq G(\omega),$$

which in turn implies $F \in \overline{\mathcal{C}_c}$. Therefore, it remains to prove that $\overline{\mathcal{C}_c} \supseteq \overline{\mathcal{C}_c^*}$, where $\mathcal{A} \supseteq \mathcal{B}$ stands for “ \mathcal{A} dominates \mathcal{B} ”. Let us fix $c > 0$. Our proof is by transfinite induction. The basis of induction $\mathcal{C}_c^0 \supseteq \mathcal{C}_c^{*0}$ follows from the fact that $\mathcal{K}_2^{G,c}$ is always dominated by $\mathcal{K}_1^{G,c}$: indeed, for a fixed $\omega \in \Omega$, any simple capital process $\mathcal{K}_t^{G,c}(\omega\xi)$ over Ω^* is a supermartingale over $t \in [1, 2]$ (see the beginning of the proof of Lemma 6.4 in [11]), where $\xi \sim W$, as above. It remains to prove that $\mathcal{C}_c^\alpha \supseteq \mathcal{C}_c^{*\alpha}$ for each countable ordinal $\alpha > 0$ assuming $\mathcal{C}_c^\beta \supseteq \mathcal{C}_c^{*\beta}$ for each $\beta < \alpha$. Let us make this assumption and let $G \in \mathcal{C}_c^{*\alpha}$. Find a sequence of functionals $G_n \in \mathcal{C}_c^{*<\alpha}$, $n = 1, 2, \dots$, such that $G \leq \liminf_{n \rightarrow \infty} G_n$. By the inductive assumption, for each n there is $G'_n \in \mathcal{C}_c^{<\alpha}$ that dominates G . By the Fatou lemma we now have, for each $\omega \in \Omega$,

$$\begin{aligned} \mathbb{E}_{W(d\xi)}(G(\omega\xi)) &\leq \mathbb{E}_{W(d\xi)}(\liminf_{n \rightarrow \infty} G_n(\omega\xi)) \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E}_{W(d\xi)}(G_n(\omega\xi)) \leq \liminf_{n \rightarrow \infty} G'_n(\omega). \end{aligned}$$

In other words, $G' := \liminf_{n \rightarrow \infty} G'_n \in \mathcal{C}_c^\alpha$ dominates G . This completes the proof of $\overline{\mathbb{E}^g}(F) \leq \overline{\mathbb{E}^g}(F^*)$.

To check that $\overline{\mathbb{E}^m}(F) \geq \overline{\mathbb{E}^m}(F^*)$, i.e.,

$$\sup_P \int F dP \geq \sup_{P^*} \int F^* dP^*,$$

where P ranges over the martingale measures on Ω and P^* over the martingale measures on Ω^* , it suffices to notice that for any P^* we can take as P the martingale measure defined by

$$P(E) := P^*(\{\omega \in \Omega^* \mid \omega_{[0,1]} \in E\})$$

for all measurable $E \subseteq \Omega$ (essentially, the restriction of P^* to cylinder sets in Ω^*).

From now on F is assumed bounded and lower semicontinuous in the Hausdorff metric.

3.2.2 Reductions II

We further simplify the functional F analogously to the series of reductions in [11], Section 10. We will modify the notation of [11] and write $\tilde{\omega}$ for $\text{ntt}(\omega)$ (as defined in Section 5 of [11]) and ϕ_s for τ_s (also defined in Section 5 of [11]). Let the domain of $\tilde{\omega}$ be $[0, D(\omega)]$ or $[0, D(\omega))$ (it has this form for typical $\omega \in \Omega$).

Let Ω'' be the family of all sets of the form $A \cup \{1\} \times [0, \infty)$ where $A \subseteq [0, 1] \times [0, \infty)$ is a bounded closed set and $\Omega' \subseteq \Omega''$ be the set of all $A \in \Omega''$ satisfying

- each vertical cut $A^t := \{a \mid (t, a) \in A\} \subseteq [0, \infty)$ of A , where $t \in [0, 1]$, is non-empty and connected (i.e., is a closed interval);
- $A^0 \ni 1$ (and, automatically, $A^1 = [0, \infty)$).

Lemma 8. *The set Ω' is closed in Ω'' (equipped with the Hausdorff metric).*

Proof. Let $A_n \rightarrow A$ for some $A_n \in \Omega'$, $n = 1, 2, \dots$, and $A \in \Omega''$; our goal is to prove $A \in \Omega'$. Let $B > 0$ be such that $A \subseteq [0, 1] \times [0, B] \cup \{1\} \times [0, \infty)$. First we check that each cut of A is non-empty: indeed, suppose $A^t = \emptyset$ for $t \in (0, 1)$ (the case $t \in \{0, 1\}$ is trivial); since $[0, B]$ is compact, this implies $A^{t'} = \emptyset$ for all t' in a neighbourhood of t ; therefore, $(A')^t = \emptyset$ for all A' in a Hausdorff neighbourhood of A . Now suppose there is $t \in [0, 1]$ (the case $t = 0$ will be also covered by our argument) such that A^t is not connected, say A^t contains points both above and below $b \in (0, B) \setminus A^t$. Let O be a connected open neighbourhood of t and δ be a strictly positive constant such that, for all $s \in O$, A^s contains points below $b - \delta$ or points above $b + \delta$ but does not contain points in $(b - \delta, b + \delta)$. Choose another connected open neighbourhood O' of t such that $\overline{O'} \subseteq O$. Let O_n^+ be the set of $s \in O'$ such that A_n^s contains points above b and O_n^- be the set of $s \in O'$ such that A_n^s contains points below b . Since, for sufficiently large n , O_n^+ and O_n^- are disjoint sets that are closed in O' (closed in O' by the compactness of $[0, b]$ and $[b, B]$) and O' is connected, either $O' = O_n^+$ or $O' = O_n^-$. This makes $A_n \rightarrow A$ impossible. The remaining condition, $A^0 \ni 1$, is obvious. \square

Now it is easy to see that Ω' is the closure of $\bar{\Omega} := \{\tilde{\omega} \mid \omega \in \Omega\}$ in Ω'' .

We extend the functional F to the set Ω' by setting

$$F'(A) := \liminf_{\omega \rightsquigarrow A} F(\omega),$$

where ω ranges over Ω and $\omega \rightsquigarrow A$ is the convergence in the sense of the “one-sided Hausdorff metric” (defined in terms of ℓ_∞ , as always in this paper): namely, the ϵ -neighbourhood of A is the set of $\omega \in \Omega$ such that

$$\sup_{(t,a) \in \text{graph}(\omega)} \inf_{(t',a') \in A} |t - t'| \vee |a - a'| < \epsilon,$$

and $\liminf_{\omega \rightsquigarrow A} F(\omega)$ is the limit of the infimum of F over the ϵ -neighbourhood of A as $\epsilon \rightarrow 0$. Since no ϵ -neighbourhood of $A \in \Omega'$ is empty for $\epsilon > 0$ (see Lemma 9 below), F' takes values in $[0, \sup F]$. Notice that F' is monotonic: $F'(A) \geq F'(B)$ when $A \subseteq B$.

Lemma 9. *Let $A \in \Omega'$ and $\epsilon > 0$. The ϵ -neighbourhood of A is not empty.*

Proof. Draw parallel vertical lines $t = i/n$, $i = 0, \dots, n$, at regular intervals in the semi-infinite region $[0, 1] \times [0, \infty)$ of the (t, a) -plane starting from $t = 0$ and ending at $t = 1$; the interval $1/n$ between the lines should be at most ϵ : $1/n \leq \epsilon$. Similarly, draw parallel horizontal lines $a = i/n$, $i = 0, 1, \dots$, at regular intervals in the same semi-infinite region $[0, 1] \times [0, \infty)$ starting from $a = 0$. The region $[0, 1] \times [0, \infty)$ will be split into squares of size at most $\epsilon \times \epsilon$; these squares can be partitioned into columns (each column consisting of squares with equal t -coordinates). Let us mark the squares whose intersection with A is non-empty. It suffices to prove that in each column the marked squares form a contiguous array and that these arrays overlap for each pair of adjacent columns: indeed, in this case we will be able to travel in a continuous manner from the point $(0, 1)$ to the line $t = 1$ via marked squares.

Suppose there is an unmarked square such that there is a point $(t', a') \in A$ in a square below it (in the same column) and there is a point $(t'', a'') \in A$ in a square above it (in the same column). (Notice that this unmarked square cannot be in the right-most column, and so the column containing the unmarked square can be regarded as bounded since A is bounded, apart from the line $t = 1$.) Suppose, for concreteness, $t' < t''$. All $t \in [t', t'']$ are now split into two disjoint closed sets: those for which there are $(t, a) \in A$ for a above the unmarked square and those for which there are $(t, a) \in A$ for a below the unmarked square. Since $[t', t'']$ is connected, one of those disjoint closed sets is empty, and we have arrived at a contradiction.

Now it is obvious that the arrays of marked squares overlap for each pair of adjacent columns: remember that the intersection of A with the vertical line between the two columns is non-empty and connected. \square

Let us check that $F' : \Omega' \rightarrow [0, \infty)$ is lower semicontinuous and that $F'(\bar{\omega}) = F(\omega)$ for all $\omega \in \Omega$; the latter property can be written as $F'|_{\Omega} = F$, where $F'|_{\Omega} : \Omega \rightarrow [0, \infty)$ is defined by $F'|_{\Omega}(\omega) := F'(\bar{\omega})$. Indeed:

- Let $c := F'(A)$ and $\epsilon > 0$; we are required to prove that $F'(B) \geq c - \epsilon$ for all B in an open Hausdorff ball around A . Let $\delta > 0$ be so small that $F(\omega) > c - \epsilon$ for all $\omega \in \Omega$ in the δ -neighbourhood of A . Let B be in the open $\delta/2$ -ball around A (in the sense of the Hausdorff metric). If ω is in the $\delta/2$ -neighbourhood of B , then ω will be in the δ -neighbourhood of A , and so $F(\omega) > c - \epsilon$. Therefore, for such B we have $F'(B) \geq c - \epsilon$.
- Let $\omega \in \Omega$. We have $F'(\bar{\omega}) \leq F(\omega)$ since ω is in the ϵ -neighbourhood of $\bar{\omega}$ for any $\epsilon > 0$. And the inequality $F'(\bar{\omega}) \geq F(\omega)$ follows from the lower semicontinuity of F on Ω (in the metric ρ_H) and the fact that $\omega_n \rightharpoonup \bar{\omega}$ implies $\rho_H(\omega_n, \omega) \rightarrow 0$. To check the last statement, suppose that there is a subsequence of ω_n such that $\rho_H(\omega_n, \omega) \geq \epsilon$ for the subsequence, where $\epsilon > 0$; without loss of generality we can assume that for each element of the subsequence there is a point $(t_n, a_n) \in \text{graph}(\omega)$ such that $t_n \leq 1 - \epsilon$ and there are no points of $\text{graph}(\omega_n)$ in the square $[t_n - \epsilon, t_n + \epsilon] \times [a_n - \epsilon, a_n + \epsilon]$.

Let (t, a) be a limit point of (t_n, a_n) , which obviously exists and belongs to $\text{graph}(\omega)$. There is another subsequence of ω_n for which there are no points of $\text{graph}(\omega_n)$ in the square $[t - \epsilon/2, t + \epsilon/2] \times [a - \epsilon/2, a + \epsilon/2]$. This contradicts $\omega_n \rightharpoonup \bar{\omega}$: the distance from $(t, \omega_n(t))$ to any point of $\text{graph}(\omega)$ stays above a strictly positive constant as $n \rightarrow \infty$.

Let us now check that we can assume $F = F'|_\Omega$ where $F' : \Omega' \rightarrow [0, \infty)$ is continuous (in the Hausdorff metric). First suppose (13) holds for the restrictions to Ω of all continuous functions of the type $\Omega' \rightarrow [0, \infty)$, but we are given $F = F'|_\Omega$ for F' that is only lower semicontinuous. Each lower semicontinuous function on a metric space (such as Ω' with the Hausdorff metric) is the limit of an increasing sequence of continuous functions (see, e.g., [4], 1.7.15(c)), so we can find an increasing sequence of continuous functionals $F_n \nearrow F'$ on Ω' . Let $\epsilon > 0$. For each n , by assumption we have $F_n|_\Omega \in \overline{\mathcal{C}_c}$ where $c := \overline{\mathbb{E}^m}(F) + \epsilon > \overline{\mathbb{E}^m}(F_n|_\Omega)$. Since $\overline{\mathcal{C}_c}$ is lim inf-closed,

$$F = \liminf_{n \rightarrow \infty} F_n|_\Omega \in \overline{\mathcal{C}_c}.$$

Therefore, $\overline{\mathbb{E}^g}(F) \leq c = \overline{\mathbb{E}^m}(F) + \epsilon$ and so, since ϵ can be arbitrarily small, $\overline{\mathbb{E}^g}(F) \leq \overline{\mathbb{E}^m}(F)$.

Let us check that we can replace our new assumption of continuity by the assumption that F depends on $\omega \in \Omega$ only via the values $\tilde{\omega}(iS/N)$ and $\phi_{iS/N}(\omega)$, $i = 1, \dots, N$ (remember that we are interested in the case $\tilde{\omega}(0) = \omega(0) = 1$), for some $S > 0$ and some $N \in \mathbb{N}$ (in particular, only via $\tilde{\omega}|_{[0, S]}$ and $\phi(\omega)|_{[0, S]}$). We ignore events of zero upper game-theoretic probability (such as the event that $\tilde{\omega}$ does not exist). Let $\epsilon > 0$ and let S and N be sufficiently large (we will explain later how large S and N should be for a given ϵ). Let $A_1 \subseteq \Omega$ consist of all $\omega \in \Omega$ such that $D(\omega) > S$ ($D(\omega)$ is defined at the beginning of this subsection on p. 10). Take S so large that the probability that a Brownian motion started from 1 at time 0 is positive over the time interval $[0, S]$ is less than ϵ .

Let $\mathfrak{K} \subseteq C_1[0, S]$ be a compact set whose Wiener measure (the distribution of a Brownian motion W^1 on $C[0, S]$ starting from 1) is more than $1 - \epsilon$. Let f be the optimal modulus of continuity for all $\psi \in \mathfrak{K}$:

$$f(\delta) := \sup_{\substack{(t_1, t_2) \in [0, S]^2: |t_1 - t_2| \leq \delta, \\ \psi \in \mathfrak{K}}} |\psi(t_1) - \psi(t_2)|, \quad \delta > 0;$$

f is an increasing function, $f(a + b) \leq f(a) + f(b)$ for all $a, b \in [0, \infty)$, and we know that $\lim_{\delta \rightarrow 0} f(\delta) = 0$ (cf. the Arzelà–Ascoli theorem). Extend \mathfrak{K} by including in it all $\omega \in C_1[0, S]$ with f as a modulus of continuity; \mathfrak{K} will stay compact with $W^1(\mathfrak{K}) > 1 - \epsilon$. Let $A_2 := \{\omega \in \Omega \mid \tilde{\omega}|_{[0, S]} \notin \mathfrak{K}\}$, where $\tilde{\omega}|_{[0, S]}(t) := \tilde{\omega}(D(\omega))$ for t such that $D(\omega) \leq t \leq S$.

Set $B := 1 + f(S)$; notice that $\sup \omega \leq B$ for all $\omega \in \Omega \setminus (A_1 \cup A_2)$.

Define $D_N^{S, f} \subseteq [0, B]^N \times [0, 1]^N$ to be the set of all sequences

$$(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, B]^N \times [0, 1]^N$$

satisfying

$$\begin{cases} v_0 := 0 \leq v_1 \leq \dots \leq v_N \leq v_{N+1} := 1, \\ |x_j - x_i| \leq f((j-i)S/N) \text{ for all } i, j \in \{0, \dots, N\} \text{ such that } i < j, \end{cases} \quad (15)$$

where $x_0 := 1$ (notice that we do not require $v_i < v_{i+1}$ when $v_{i+1} < 1$, in order to make the set (15) closed). Define a function $U_N^{S,f} : D_N^{S,f} \rightarrow [0, \sup F]$ by

$$U_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N) := F' \left(A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N) \right), \quad (16)$$

where F' is the continuous function on Ω' defined earlier and the set $A := A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N) \in \Omega'$ is defined by the following conditions:

- for all $i \in \{0, \dots, N\}$ and $t \in (v_i, v_{i+1})$,

$$A^t = [x_i \wedge x_{i+1} - f(S/N), x_i \vee x_{i+1} + f(S/N)]$$

(with $x_i \wedge x_{i+1} = x_i \vee x_{i+1} := x_N$ when $i = N$);

- for all $i, j \in \{0, \dots, N\}$ such that $i < j$ and $v_i < t := v_{i+1} = v_{i+2} = \dots = v_j < v_{j+1}$,

$$A^t = \left[\bigwedge_{k=i}^{j+1} x_k - f(S/N), \bigvee_{k=i}^{j+1} x_k + f(S/N) \right];$$

- $A^0 = [1 \wedge x_1 - f(S/N), 1 \vee x_1 + f(S/N)]$;
- $A^1 = [0, \infty)$.

Therefore, A consists of a sequence of horizontal slabs of width at least $2f(S/N)$ separated by vertical lines. This set contains $\{1\} \times [0, \infty)$ and, for all $i = 0, \dots, N$, also contains (v_i, x_i) .

The metric on $D_N^{S,f}$ is defined by

$$\begin{aligned} \rho((x_1, \dots, x_N; v_1, \dots, v_N), (x'_1, \dots, x'_N; v'_1, \dots, v'_N)) \\ := \bigvee_{j=1}^N \rho_H((v_j, x_j), (v'_j, x'_j)), \end{aligned} \quad (17)$$

where the metric ρ_H on $[0, 1] \times [0, \infty)$ is defined by

$$\begin{aligned} \rho_H((v, x), (v', x')) &:= H(\{(v, x)\} \cup \{1\} \times [0, \infty), \{(v', x')\} \cup \{1\} \times [0, \infty)) \\ &= (|v - v'| \vee |x - x'|) \wedge (1 - v \wedge v'), \end{aligned} \quad (18)$$

H standing for the Hausdorff metric defined in terms of the ℓ_∞ metric on $[0, 1] \times [0, \infty)$, as before.

Lemma 10. *Each function $U_N^{S,f}$ is continuous on $D_N^{S,f}$ under our definition (16) and the metric (17).*

Proof. Fix some $(x_1, \dots, x_N; v_1, \dots, v_N) \in D_N^{S,f}$. Let $(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \in D_N^{S,f}$ for $n = 1, 2, \dots$ and $(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \rightarrow (x_1, \dots, x_N; v_1, \dots, v_N)$ in ρ as $n \rightarrow \infty$. It is easy to see that, in the Hausdorff metric,

$$A_N^{S,f}(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \rightarrow A_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N)$$

as $n \rightarrow \infty$. This implies

$$U_N^{S,f}(x_1^n, \dots, x_N^n; v_1^n, \dots, v_N^n) \rightarrow U_N^{S,f}(x_1, \dots, x_N; v_1, \dots, v_N)$$

as $n \rightarrow \infty$ and completes the proof. \square

Define a functional $F_N^{S,f} : \Omega \rightarrow [0, \sup F]$ by

$$F_N^{S,f}(\omega) = U_N^{S,f}(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1), \quad \omega \in \Omega; \quad (19)$$

when the argument on the right-hand side is outside the domain $D_N^{S,f}$ of $U_N^{S,f}$, set $F_N^{S,f}(\omega) := \sup F$.

The following lemma lists the main properties of the sequence of functionals $F_N^{S,f}$, $N = 1, 2, \dots$, that we will need.

Lemma 11. *For all $\omega \in \Omega \setminus (A_1 \cup A_2)$,*

$$\forall N : F_N^{S,f}(\omega) \leq F(\omega)$$

and

$$\liminf_{N \rightarrow \infty} F_N^{S,f}(\omega) \geq F(\omega). \quad (20)$$

Proof. Notice that $\omega \notin A_1$ implies $\phi_S(\omega) = 1$; therefore, $\omega \in \Omega \setminus (A_1 \cup A_2)$ implies

$$\bar{\omega} \subseteq A_N^{S,f}(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1),$$

which immediately implies $F_N^{S,f}(\omega) \leq F(\omega)$. Since for $\omega \in \Omega \setminus (A_1 \cup A_2)$ the Hausdorff distance between

$$A_N^{S,f}(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1)$$

and $\bar{\omega}$ tends to 0 as $N \rightarrow \infty$, we also have (20). \square

Let us extend $U_N^{S,f}$ to the whole of

$$\{(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, B]^N \times [0, 1]^N \mid v_1 \leq \dots \leq v_N\}$$

obtaining a continuous function \tilde{U}_N taking values in $[0, \sup F]$; this is possible by the Tietze–Urysohn theorem (see, e.g., [4], 2.1.8). Since the domain of the function \tilde{U}_N is compact (in the usual topology, let alone in the topology generated by ρ), this function is uniformly continuous. Finally, extend \tilde{U}_N to the whole of

$$D_N := \{(x_1, \dots, x_N; v_1, \dots, v_N) \in [0, \infty)^N \times [0, 1]^N \mid v_1 \leq \dots \leq v_N\}$$

by

$$U_N(x_1, \dots, x_N; v_1, \dots, v_N) := \tilde{U}_N(x_1 \wedge B, \dots, x_N \wedge B; v_1, \dots, v_N).$$

The function U_N inherits the uniform continuity of \tilde{U}_N .

Analogously to (19), define a functional F_N by

$$\begin{aligned} F_N(\omega) = U_N(\omega(\phi_{S/N}(\omega) \wedge 1), \omega(\phi_{2S/N}(\omega) \wedge 1), \dots, \omega(\phi_S(\omega) \wedge 1); \\ \phi_{S/N}(\omega) \wedge 1, \phi_{2S/N}(\omega) \wedge 1, \dots, \phi_S(\omega) \wedge 1); \end{aligned} \quad (21)$$

by the definition of U_N , $F_N(\omega) = F_N^{S,f}(\omega)$ when $\omega \in \Omega \setminus (A_1 \cup A_2)$.

Our task is now reduced to proving $\overline{\mathbb{E}}^g(F_N) \leq \overline{\mathbb{E}}^m(F_N)$. To demonstrate this, we first notice that

$$\overline{\mathbb{P}}^g(A_1) \leq \epsilon, \quad \overline{\mathbb{P}}^g(A_2) \leq \epsilon, \quad (22)$$

$$\overline{\mathbb{P}}^m(A_1) \leq \overline{\mathbb{P}}^g(A_1) \leq \epsilon, \quad \overline{\mathbb{P}}^m(A_2) \leq \overline{\mathbb{P}}^g(A_2) \leq \epsilon; \quad (23)$$

indeed, (22) follows from Theorem 3.1 of [11] and the time-superinvariance of the sets

$$\{\omega \in C_1[0, \infty) \mid \tilde{\omega} \text{ is defined and positive over } [0, S]\}$$

and

$$\{\omega \in C_1[0, \infty) \mid \tilde{\omega} \text{ is defined over } [0, S] \text{ and } \tilde{\omega}|_{[0, S]} \notin \mathfrak{R}\},$$

and (23) follows from $\overline{\mathbb{P}}^m \leq \overline{\mathbb{P}}^g$, established in the previous subsection: see (10). In combination with Lemmas 3, 5, 11, and the assumption $\overline{\mathbb{E}}^g(F_N) \leq \overline{\mathbb{E}}^m(F_N)$, for all N , this implies

$$\begin{aligned} \overline{\mathbb{E}}^g(F) &\leq \overline{\mathbb{E}}^g\left(\liminf_{N \rightarrow \infty} F_N^{S,f}\right) + 2C\epsilon \leq \liminf_{N \rightarrow \infty} \overline{\mathbb{E}}^g(F_N^{S,f}) + 2C\epsilon \\ &\leq \liminf_{N \rightarrow \infty} \overline{\mathbb{E}}^g(F_N) + 4C\epsilon \leq \liminf_{N \rightarrow \infty} \overline{\mathbb{E}}^m(F_N) + 4C\epsilon \\ &\leq \liminf_{N \rightarrow \infty} \overline{\mathbb{E}}^m(F_N^{S,f}) + 6C\epsilon \leq \overline{\mathbb{E}}^m(F) + 8C\epsilon \end{aligned}$$

for $C := \sup F$. Since ϵ can be arbitrarily small, this achieves our goal.

3.2.3 Setting intermediate goals

Let us fix S and N ; our goal is to prove $\overline{\mathbb{E}}^g(F_N) \leq \overline{\mathbb{E}}^m(F_N)$. We will abbreviate U_N to U .

We start the proof by defining functions

$$\begin{aligned} U_i^e : D_i^e &\rightarrow [0, \infty), \quad i = 0, \dots, N, \\ U_i^m : D_i^m &\rightarrow [0, \infty), \quad i = 0, \dots, N-1 \end{aligned}$$

(with “m” standing for “maximization” and “e” for “expectation”) whose domains are

$$\begin{aligned} D_i^e &:= \left\{ (x_1, v_1, \dots, x_i, v_i) \in ([0, \infty) \times [0, 1])^i \mid \right. \\ &\quad \left. v_1 \leq \dots \leq v_i \text{ and } (x_j = x_{j+1} \text{ whenever } j < i \text{ and } v_j = 1) \right\}, \\ D_i^m &:= \left\{ (x_1, v_1, \dots, x_i, v_i, x_{i+1}) \in ([0, \infty) \times [0, 1])^i \times [0, \infty) \mid \right. \\ &\quad \left. v_1 \leq \dots \leq v_i \text{ and } (x_j = x_{j+1} \text{ whenever } j \leq i \text{ and } v_j = 1) \right\}. \end{aligned}$$

They will be defined by induction in i .

The basis of induction is

$$U_N^e(x_1, v_1, \dots, x_N, v_N) := U(x_1, \dots, x_N; v_1, \dots, v_N). \quad (24)$$

Given U_{i+1}^e , where $i := N-1$, we define

$$U_i^m(x_1, v_1, \dots, x_i, v_i, x_{i+1}) := \sup_{v \in [v_i, 1]} U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v). \quad (25)$$

Given U_i^m , where $i := N-1$, we next define

$$U_i^e(x_1, v_1, \dots, x_i, v_i) = \begin{cases} U_i^m(x_1, v_1, \dots, x_i, v_i, x_i) & \text{if } v_i = 1 \\ \mathbb{E} U_i^m(x_1, v_1, \dots, x_i, v_i, \xi) & \text{otherwise} \end{cases} \quad (26)$$

where $\xi \geq 0$ is the value at time S/N of a linear Brownian motion that starts at x_i at time 0 and is stopped when it hits level 0. Next use alternately (25) and (26) for

$$i = N-2, N-2; N-3, N-3; \dots; 1, 1$$

to define inductively other U_i^m and U_i^e . Finally, define

$$U_0^m(x_1) := \sup_{v \in [0, 1]} U_1^e(x_1, v), \quad U_0^e := \mathbb{E} U_0^m(\xi)$$

where $\xi \geq 0$ is the value at time S/N of a linear Brownian motion that starts at 1 at time 0 and is stopped when it hits level 0 (the last event being unlikely for a large N).

In this proof we will show that U_0^e is sandwiched between $\overline{\mathbb{E}}^m(F_N)$ and $\overline{\mathbb{E}}^g(F_N)$ as $\overline{\mathbb{E}}^m(F_N) \geq U_0^e \geq \overline{\mathbb{E}}^g(F_N)$, which will achieve our goal. But first we discuss some properties of regularity of the intermediate functions U_i^m and U_i^e .

It is obvious that each of the functions U_i^e and U_i^m is bounded (by $\sup F$), and the following two lemmas imply that they are uniformly continuous. The metric on D_i^e is defined by

$$\rho^e((x_1, v_1, \dots, x_i, v_i), (x'_1, v'_1, \dots, x'_i, v'_i)) := \bigvee_{j=1}^i \rho_H((v_j, x_j), (v'_j, x'_j)),$$

ρ_H being defined in (18). The metric on D_i^m is defined by

$$\begin{aligned} \rho^m((x_1, v_1, \dots, x_i, v_i, x_{i+1}), (x'_1, v'_1, \dots, x'_i, v'_i, x'_{i+1})) \\ := \sup_{v \in [v_i \wedge v'_i, 1]} \rho^e((x_1, v_1, \dots, x_i, v_i, x_{i+1}, v \vee v_i), \\ (x'_1, v'_1, \dots, x'_i, v'_i, x'_{i+1}, v \vee v'_i)). \end{aligned}$$

Lemma 12. *If a function U_{i+1}^e on D_{i+1}^e is uniformly continuous, then the function U_i^m on D_i^m defined by (25) is also uniformly continuous (with the same modulus of continuity).*

Proof. Let f be a modulus of continuity for U_{i+1}^e (in this paper we only consider increasing moduli of continuity). It suffices to prove that, for each $\delta > 0$,

$$\begin{aligned} \sup_{v \in [v_i, 1]} U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v) \\ \geq \sup_{v \in [v'_i, 1]} U_{i+1}^e(x'_1, v'_1, \dots, x'_i, v'_i, x'_{i+1}, v) - f(\delta) \end{aligned} \quad (27)$$

provided the D_i^m distance between $(x_1, v_1, \dots, x_{i+1})$ and $(x'_1, v'_1, \dots, x'_{i+1})$ does not exceed δ . This follows from

$$\begin{aligned} \sup_{v \in [v_i, 1]} U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v) &\geq U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v' \vee v_i) \\ &\geq U_{i+1}^e(x'_1, v'_1, \dots, x'_i, v'_i, x'_{i+1}, v') - f(\delta) \\ &= \sup_{v \in [v'_i, 1]} U_{i+1}^e(x'_1, v'_1, \dots, x'_i, v'_i, x'_{i+1}, v) - f(\delta), \end{aligned}$$

where $v' \geq v'_i$ is the point at which the supremum on the right-hand side of (27) is attained. \square

Lemma 13. *If a function U_i^m on D_i^m is bounded and uniformly continuous, then the function U_i^e on D_i^e defined by (26) is also uniformly continuous.*

Proof. Let $\delta > 0$ and f be the optimal modulus of continuity for U_i^m ; to bound the optimal modulus of continuity for U_i^e , we consider three possibilities for two points E and E' in D_i^e where $E = (x_1, v_1, \dots, x_i, v_i)$, $E' = (x'_1, v'_1, \dots, x'_i, v'_i)$, and $\rho^e(E, E') \leq \delta$.

- If $v_i = v'_i = 1$, the difference between $U_i^e(E)$ and $U_i^e(E')$ does not exceed $f(\delta)$.
- If $v_i < v'_i = 1$ or $v'_i < v_i = 1$, the difference between $U_i^e(E)$ and $U_i^e(E')$ also does not exceed $f(\delta)$. Indeed, suppose, for concreteness, that $v'_i = 1$. Then $v_i \geq 1 - \delta$. By definition, $U_i^e(E)$ is an average of $U_i^m(E, x)$ over x , and $U_i^e(E')$ coincides with $U_i^m(E', x'_i)$. By the definition of the metric on D_i^m , the ρ^m distance between (E, x) and (E', x'_i) is at most δ , and so the difference between $U_i^e(E)$ and $U_i^e(E')$ does not exceed $f(\delta)$.

- If $v_i < 1$ and $v'_i < 1$, the difference between $U_i^e(E)$ and $U_i^e(E')$ does not exceed $2f(\delta) + C\delta\sqrt{N/S}$, where C is an upper bound on U_i^m . Let us check this. Our goal is to prove that

$$|\mathbb{E} U_i^m(E, \xi) - \mathbb{E} U_i^m(E', \xi')| \leq 2f(\delta) + C\delta\sqrt{N/S}$$

where ξ (resp. ξ') is the value at time S/N of a linear Brownian motion that starts at x_i (resp. x'_i) at time 0 and is stopped when it hits level 0. It suffices to notice that

$$\begin{aligned} & |\mathbb{E} U_i^m(E, \xi) - \mathbb{E} U_i^m(E', \xi')| \\ & \leq |\mathbb{E} U_i^m(E, \xi) - \mathbb{E} U_i^m(E, \xi')| + |\mathbb{E} U_i^m(E, \xi') - \mathbb{E} U_i^m(E', \xi')| \quad (28) \\ & \leq f(\delta) + C\delta/\sqrt{S/N} + f(\delta). \end{aligned}$$

The upper bound $f(\delta) + C\delta/\sqrt{S/N}$ on the first addend in (28) follows from Lemma 14 below; we also used the uniform continuity of $U_i^m(E, \cdot)$ and $U_i^m(\cdot, x)$, where $x \in [0, \infty)$, with f as modulus of continuity.

In all three cases the difference is bounded by $2f(\delta) + C\delta\sqrt{N/S}$. \square

The following result was used in the proof of Lemma 13 above.

Lemma 14. *Suppose $a > 0$ and $u : [0, \infty) \rightarrow [0, C]$ is a bounded uniformly continuous function with f as modulus of continuity. Then*

$$x \in [0, \infty) \mapsto \mathbb{E} u(W_{\tau \wedge a}^x),$$

where W^x is a Brownian motion started at x and τ is the moment it hits level 0, is uniformly continuous with $\delta > 0 \mapsto f(\delta) + C\delta/\sqrt{a}$ as modulus of continuity.

Proof. Consider points $x \in [0, \infty)$ and $x' \in (x, x + \delta]$, for some $\delta > 0$. Let us map each path of $W_{\tau \wedge a}^{x'}$ to the path of $W_{\tau \wedge a}^x$ obtained by subtracting $x' - x$ and stopping when level 0 is hit; we will refer to the latter as the path *corresponding* to the former. There are three kinds of paths of $W_{\tau \wedge a}^{x'}$:

- Those that never hit level $x' - x$ over the time interval $[0, a]$. The average of $u(W_{\tau \wedge a}^{x'}) = u(W_a^{x'})$ over such paths and the average of $u(W_{\tau \wedge a}^x) = u(W_a^x)$ over the corresponding paths differ by at most $f(\delta)$.
- Those that hit level 0 over $[0, a]$. The average of $u(W_{\tau \wedge a}^{x'}) = u(0)$ over such paths and the average of $u(W_{\tau \wedge a}^x) = u(0)$ over the corresponding paths coincide.
- Those that hit level $x' - x$ but never hit level 0 over $[0, a]$. The probability of such paths is

$$\begin{aligned} 2\Phi(-x/\sqrt{a}) - 2\Phi(-x'/\sqrt{a}) & \leq 2\mathbb{P}(\xi \in [0, (x' - x)/\sqrt{a}]) \\ & < \frac{2}{\sqrt{2\pi}}(x' - x)/\sqrt{a} < \delta/\sqrt{a}, \end{aligned}$$

where Φ is the standard normal distribution function, $\xi \sim \Phi$, and the factor of 2 comes from the reflection principle.

Therefore, the overall averages of $u(W_{\tau \wedge a}^x)$ and $u(W_{\tau \wedge a}^{x'})$ differ by at most $f(\delta) + C\delta/\sqrt{a}$. \square

3.2.4 Tackling measure-theoretic probability

First we prove an easy auxiliary statement ensuring the existence of measurable “choice functions”.

Lemma 15. *Suppose $\{A_\theta \mid \theta \in \Theta\}$ is a countable cover of a measurable space Ω such that each A_θ is measurable. There is a measurable function $f : \Omega \rightarrow \Theta$ (with the discrete σ -algebra on Θ) such that $\omega \in A_{f(\omega)}$ for all $\omega \in \Omega$.*

Proof. Assume, without loss of generality, $\Theta = \mathbb{N}$. Define

$$f(\omega) := \min\{\theta \mid \omega \in A_\theta\}.$$

Then, for each $\theta \in \mathbb{N}$, the set

$$\{\omega \mid f(\omega) \leq \theta\} = A_1 \cup \dots \cup A_\theta$$

is measurable. \square

In this section we show that $\bar{\mathbb{E}}^m(F_N) \geq U_0^s$. We define a martingale measure P by backward induction. For each $i = 0, \dots, N-1$, let V_{i+1} be a Borel function on D_i^m such that, for all $(x_1, v_1, \dots, x_i, v_i, x_{i+1}) \in D_i^m$ satisfying $v_i < 1$, it is true that

$$v_i < V_{i+1}(x_1, v_1, \dots, x_i, v_i, x_{i+1}) < 1$$

and

$$\begin{aligned} U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, V_{i+1}(x_1, v_1, \dots, x_i, v_i, x_{i+1})) \\ \geq U_i^m(x_1, v_1, \dots, x_i, v_i, x_{i+1}) - \epsilon \end{aligned}$$

(cf. (25)), where $\epsilon > 0$ is a small constant (further details will be added later). (Intuitively, V_{i+1} outputs a $v > v_i$ at which the supremum of $U_{i+1}^e(x_1, v_1, \dots, x_{i+1}, v)$ is almost attained.) The existence of such V_{i+1} follows from Lemma 15: indeed, for each rational $r \in (0, 1)$ the set

$$\begin{aligned} A_r := \{(x_1, v_1, \dots, x_i, v_i, x_{i+1}) \in D_i^m \mid r > v_i \text{ and} \\ U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, r) \geq U_i^m(x_1, v_1, \dots, x_i, v_i, x_{i+1}) - \epsilon\} \end{aligned}$$

is Borel (namely, intersection of open and closed), and the sets A_r form a cover of D_i^m . By the uniform continuity of U_{i+1}^e and U_i^m , there is $\delta > 0$ such that, for all i (remember that there are finitely many i) and for all $x_1, v_1, \dots, x_i, v_i, x_{i+1}$, and x'_{i+1} ,

$$\begin{aligned} |x'_{i+1} - x_{i+1}| < \delta \\ \implies U_{i+1}^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, V_{i+1}(x_1, v_1, \dots, x_i, v_i, x'_{i+1})) \end{aligned}$$

$$\geq U_i^m(x_1, v_1, \dots, x_i, v_i, x'_{i+1}) - 2\epsilon. \quad (29)$$

Next choose Borel V_i^* such that, for $v_i < 1$,

$$V_{i+1}(x_1, v_1, \dots, x_i, v_i, \xi) > V_i^*(x_1, v_1, \dots, x_i, v_i) > v_i \quad (30)$$

with probability (over ξ only) at least $1 - \epsilon$ when ξ is the value taken at time S/N by a linear Brownian motion started from x_i at time 0 and stopped when it hits level 0. (The existence of V_i^* also follows from Lemma 15.) Let $\Delta \in (0, S/N)$ be such that

$$\sup_{t \in [0, \Delta]} |W_t| < \delta \quad (31)$$

with a probability at least $1 - \epsilon$, where W is a standard Brownian motion.

By a *scaled Brownian motion* we will mean a process of the type W_{ct} where W is a Brownian motion and $c > 0$ (equivalently, a process of the type cW_t where W is a Brownian motion and $c > 0$). Define a probability measure P on Ω as the distribution of $\omega \in \Omega$ generated as follows. For $i = 0, 1, \dots, N - 1$:

- Start a scaled Brownian motion W^i (independent of what has happened before if $i > 0$) from x_i (with $x_0 := 1$) at time v_i (with $v_0 := 0$) such that its quadratic variation over $[v_i, v_i^*]$ is $S/N - \Delta$, where

$$v_i^* := V_i^*(x_1, v_1, \dots, x_i, v_i) < 1.$$

Define

$$\omega|_{[v_i, v_i^*]} := W^{\circ, i}|_{[v_i, v_i^*]}$$

where $W^{\circ, i}$ is W^i stopped when it hits level 0. If $\omega(v_i^*) = 0$, the random process of generating ω is complete; set $\omega|_{[v_i^*, 1]} := 0$, $v_{i+1}^* = \dots = v_{N-1}^* := 1$, and $v_{i+1} = \dots = v_N := 1$, and then stop.

- Set

$$v_{i+1} := \begin{cases} V_{i+1}(x_1, v_1, \dots, x_i, v_i, \omega(v_i^*)) & \text{if } V_{i+1}(x_1, v_1, \dots, x_i, v_i, \omega(v_i^*)) > v_i^* \\ 1 & \text{otherwise.} \end{cases}$$

Start another independent Brownian motion \bar{W}^i from $\omega(v_i^*)$ at time v_i^* such that its quadratic variation over $[v_i^*, v_{i+1}]$ is Δ . Define

$$\omega|_{[v_i^*, v_{i+1}]} := \bar{W}^{\circ, i}|_{[v_i^*, v_{i+1}]}$$

where $\bar{W}^{\circ, i}$ is \bar{W}^i stopped when it hits level 0. If $\omega(v_{i+1}) = 0$ or $v_{i+1} = 1$ (or both), the random process of generating ω is complete; set $\omega|_{[v_{i+1}, 1]} := 0$ if $v_{i+1} < 1$, set $v_{i+1}^* = \dots = v_{N-1}^* := 1$ and $v_{i+2} = \dots = v_N := 1$, and then stop.

- Set $x_{i+1} := \omega(v_{i+1})$; notice that $v_{i+1} < 1$.

If the procedure was not stopped, and so $v_N < 1$, define $\omega|_{[v_N, 1]}$ to be the constant $x_N = \omega(v_N)$.

Let us now check that $\mathbb{E}_P(F_N) \geq U_0^e$. More precisely, we will show by induction in i that, for $i = N, \dots, 0$,

$$\mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_i}) \geq U_i^e(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i) - (N - i)(3C + 3)\epsilon \quad \text{a.s.}, \quad (32)$$

and that, for $i = N - 1, \dots, 0$,

$$\begin{aligned} \mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_i^*}) &\geq U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) \\ &\quad - (N - i)(3C + 3)\epsilon + (C + 1)\epsilon \quad \text{a.s.}, \end{aligned} \quad (33)$$

where: $C := \sup U$; \tilde{x}_j are x_j (as defined in the definition of P) considered as function of ω (it is clear that x_j can be restored given ω P -almost surely); similarly, \tilde{v}_j and \tilde{v}_j^* are v_j and v_j^* considered as functions of ω ; $\mathcal{F}_{\tilde{v}_i}$ and $\mathcal{F}_{\tilde{v}_i^*}$ are the usual σ -algebras on Ω defined as in (1) for the stopping times \tilde{v}_i and \tilde{v}_i^* . Since, ϵ can be arbitrarily small, (32) with $i = 0$ will achieve our goal.

For $i = N$, (32) holds almost surely as $U_i^e := U := U_N$ and F_N is defined by (21).

Assuming (32) with $i + 1$ in place of i , $i < N$, let us deduce (33): concentrating on the non-trivial case $\tilde{v}_i < 1$,

$$\begin{aligned} \mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_i^*}) &= \mathbb{E}_P\left(\mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_{i+1}}) | \mathcal{F}_{\tilde{v}_i^*}\right) \\ &\geq \mathbb{E}_P\left(U_{i+1}^e(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \tilde{x}_{i+1}, \tilde{v}_{i+1}) | \mathcal{F}_{\tilde{v}_i^*}\right) - (N - i - 1)(3C + 3)\epsilon \\ &\geq U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - (N - i - 1)(3C + 3)\epsilon - (2C + 2)\epsilon \\ &= U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - (N - i)(3C + 3)\epsilon + (C + 1)\epsilon \quad \text{a.s.}, \end{aligned}$$

where the second inequality follows from the fact that

$$U_{i+1}^e(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \tilde{x}_{i+1}, \tilde{v}_{i+1}) \geq U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) - 2\epsilon$$

with $\mathcal{F}_{\tilde{v}_i^*}$ -conditional probability at least $1 - 2\epsilon$ a.s. This fact in turn follows from (30) and (31) each holding with probability at least $1 - \epsilon$ (and so the conjunction of $|\omega(\tilde{v}_{i+1}) - \omega(\tilde{v}_i^*)| < \delta$ and $\tilde{v}_{i+1} < 1$ holding with $\mathcal{F}_{\tilde{v}_i^*}$ -conditional probability at least $1 - 2\epsilon$ a.s.) combined with an application of (29).

Assuming (33) let us deduce (32): again concentrating on the case $\tilde{v}_i < 1$,

$$\begin{aligned} \mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_i}) &= \mathbb{E}_P\left(\mathbb{E}_P(F_N | \mathcal{F}_{\tilde{v}_i^*}) | \mathcal{F}_{\tilde{v}_i}\right) \\ &\geq \mathbb{E}_P\left(U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \omega(\tilde{v}_i^*)) | \mathcal{F}_{\tilde{v}_i}\right) - (N - i)(3C + 3)\epsilon + (C + 1)\epsilon \\ &= \mathbb{E}_P\left(U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \xi) | \mathcal{F}_{\tilde{v}_i}\right) - (N - i)(3C + 3)\epsilon + (C + 1)\epsilon \\ &\geq U_i^e(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i) - (N - i)(3C + 3)\epsilon \quad \text{a.s.} \end{aligned}$$

where ξ is the value at time $S/N - \Delta$ (rather than S/N as in the definition of U_i^e) of a linear Brownian motion started at \tilde{x}_i at time 0 and stopped when it hits level 0, and \mathbb{E} (without a subscript) refers to averaging over ξ only. The last inequality can be derived as follows:

- Using the time period $[0, S/N - \Delta]$ in place of $[0, S/N]$ in the definition of ξ , we make an error (in the value of ξ) of at most δ with probability at least $1 - \epsilon$: cf. (31).
- This leads to an error of at most $f(\delta)$ with probability at least $1 - \epsilon$ in the expression $\mathbb{E} U_i^m(\tilde{x}_1, \tilde{v}_1, \dots, \tilde{x}_i, \tilde{v}_i, \xi)$, where f is a modulus of continuity for all U_i^m , $i = 0, \dots, N - 1$.
- Without loss of generality assume $f(\delta) \leq \epsilon$.

3.2.5 Tackling game-theoretic probability

Now we show that $\overline{\mathbb{E}}^g(F_N) \leq U_0^e$.

Let $\epsilon > 0$ be a small positive number (see below for details of how small), let L be a large positive integer (see below for details of how large depending on ϵ), and for each $i = N, N - 1, \dots, 0$, define a function

$$\overline{U}_i : \mathbb{N}_0 \times \{0, 1, \dots, L\} \times D_i^e \rightarrow [0, \infty)$$

by

$$\overline{U}_i(X, L; x_1, v_1, \dots, x_i, v_i) := U_i^m(x_1, v_1, \dots, x_i, v_i, X\sqrt{S/NL}) \quad (34)$$

and, for $j = L - 1, \dots, 1, 0$,

$$\begin{aligned} \overline{U}_i(X, j; x_1, v_1, \dots, x_i, v_i) := \\ \frac{\overline{U}_i(X - 1, j + 1; x_1, v_1, \dots, x_i, v_i) + \overline{U}_i(X + 1, j + 1; x_1, v_1, \dots, x_i, v_i)}{2}, \end{aligned} \quad (35)$$

if $X > 0$, and

$$\overline{U}_i(0, j; x_1, v_1, \dots, x_i, v_i) := \overline{U}_i(0, j + 1; x_1, v_1, \dots, x_i, v_i). \quad (36)$$

Equations (34)–(36) assume $v_i < 1$; if $v_i = 1$, set, e.g.,

$$\overline{U}_i(X, j; x_1, v_1, \dots, x_i, v_i) := U_i^m(x_1, v_1, \dots, x_i, v_i, X\sqrt{S/NL})$$

for all $j = 0, \dots, L$ (although the only interesting case for us is $v_i < 1 - \epsilon$). We will fix $i \in \{0, 1, \dots, N\}$ for a while.

Let us check that

$$U_i^e(x_1, v_1, \dots, x_i, v_i) \approx \overline{U}_i\left(\lfloor x_i/\sqrt{S/NL} \rfloor, 0; x_1, v_1, \dots, x_i, v_i\right), \quad (37)$$

assuming $v_i < 1$. This follows from the KMT theorem (Theorem 1 of Komlós, Major, and Tusnády [6]; see also [5]); we will use its following special case ([2], Theorem 1.5).

KMT theorem. *Let E_1, E_2, \dots be i.i.d. symmetric ± 1 -valued random variables. For each k , let $S_k := \sum_{i=1}^k E_i$. It is possible to construct a version of*

the sequence $(S_k)_{k \geq 0}$ and a standard Brownian motion $(B_t)_{t \geq 0}$ on the same probability space such that, for all n and all $x \geq 0$,

$$\mathbb{P} \left(\max_{k \leq n} |S_k - B_k| \geq C_1 \ln n + x \right) \leq C_2 e^{-x},$$

where C_1 and C_2 are absolute constants.

(Although for our purpose much simpler results, such as those [9] based on Skorokhod's representation, would have been sufficient.) On the left-hand side of (37) we have the average of $\bar{U}_i^m(x_1, v_1, \dots, x_i, v_i, \cdot)$ w.r. to the value of a Brownian motion at time S/N stopped when it hits level 0 and on the right-hand side of (37) we have the average of the same function w.r. to the value of a scaled simple random walk at the same time S/N stopped when it hits level 0; the scaled random walk makes steps of S/NL in time and $\sqrt{S/NL}$ in space; the Brownian motion and random walk are started from nearby points, namely x_i and $\lfloor x_i / \sqrt{S/NL} \rfloor \sqrt{S/NL}$. By the KMT theorem there are coupled versions of the Brownian motion (not stopped) and the scaled simple random walk (also not stopped) that differ by at most ϵ over $[0, S]$ with probability at least $1 - \epsilon$, provided L is large enough. (For example, we can take L large enough for x_i and $\lfloor x_i / \sqrt{S/NL} \rfloor \sqrt{S/NL}$ to be $\epsilon/2$ -close and for the precision of the KMT approximation over $[0, S]$ to be $\epsilon/2$ with probability at least $1 - \epsilon$.) The values at time S/N of the stopped Brownian motion and stopped scaled random walk can differ by more than ϵ even when their non-stopped counterparts differ by at most ϵ over $[0, S]$, but as the argument in Lemma 14 shows, the probability of this is at most $3\epsilon / \sqrt{S/N}$ (we would have $2\epsilon / \sqrt{S/N}$ if both coupled processes were Brownian motions, and replacing 2 by 3 adjusts for the discreteness of the random walk, for large L). Therefore, the difference between the two sides of (37) does not exceed

$$g(\epsilon) := f(\epsilon) + C3\epsilon / \sqrt{S/N}, \quad (38)$$

where f is a modulus of continuity of \bar{U}_i^m for all $i = 0, \dots, N-1$ and $C := \sup U$.

For $i = 1, \dots, N$, set

$$v_i = v_i(\omega) := \phi_{iS/N}(\omega) \wedge 1, \quad (39)$$

$$x_i = x_i(\omega) := \omega(v_i). \quad (40)$$

During each non-empty time interval $[v_i(\omega), v_{i+1}(\omega))$ the trader will bet at the stopping times

$$\begin{aligned} T_{i,0}(\omega) &:= \inf \left\{ t \geq v_i(\omega) \mid \omega(t) / \sqrt{S/NL} \in \mathbb{N}_0 \right\}, \\ T_{i,j}(\omega) &:= \inf \left\{ t \geq T_{i,j-1}(\omega) \mid \omega(t) / \sqrt{S/NL} \in \mathbb{N}_0, \omega(t) \neq \omega(T_{i,j-1}(\omega)) \right\}, \\ j &\in \{1, \dots, L\}, \end{aligned}$$

such that $T_{i,j}(\omega) < v_{i+1}(\omega) \wedge (1 - \epsilon)$; therefore, we are only interested in the case $j \in \{1, \dots, J_i\}$ where

$$J_i = J_i(\omega) := \max \{ j \in \{0, \dots, L\} \mid T_{i,j}(\omega) < v_{i+1}(\omega) \}$$

($J_i = L$ being a common case). Besides, the bet at the times $v_i(\omega)$ will be set to zero unless $v_i(\omega) = T_{i,0}(\omega)$. The bets at the times $T_{i,L}(\omega)$ will also be set to zero unless $T_{i,L}(\omega) = T_{i+1,0}(\omega)$.

For $j = 0, \dots, L$, set

$$X_{i,j} := \omega(T_{i,j}) / \sqrt{S/NL} \in \mathbb{N}_0.$$

The bet at time $T_{i,j}(\omega) < 1 - \epsilon$ is 0 if $X_{i,j} = 0$ or $j = L$; otherwise, it is defined in such a way that the increase of the capital over $[T_{i,j}, T_{i,j+1}]$ is typically

$$\overline{U}_i(X_{i,j+1}, j+1; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,j}, j; x_1, v_1, \dots, x_i, v_i) \quad (41)$$

(this assumes, e.g., $T_{i,j+1} \leq v_{i+1}$); namely, the bet at time $T_{i,j}$ is formally defined as

$$\frac{\overline{U}_i(X_{i,j} + 1, j+1; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,j}, j; x_1, v_1, \dots, x_i, v_i)}{\sqrt{S/NL}}. \quad (42)$$

(When $X_{i,j+1} > X_{i,j}$, the increase is (41) by the definition of the bet, and when $X_{i,j+1} < X_{i,j}$, the increase is (41) by the definition of the bet and the definition (35).)

Let us check that this strategy achieves the final value greater than or close to $F_N(\omega)$ (with high lower game-theoretic probability) starting from U_0^e . More generally, we will check that the capital \mathcal{K} of this strategy (started with U_0^e) at time $v_i(\omega)$, $i = 0, 1, \dots, N$, satisfies

$$\mathcal{K}_{v_i(\omega)} \gtrsim U_i^e(x_1(\omega), v_1(\omega), \dots, x_i(\omega), v_i(\omega))$$

with lower game-theoretic probability close to 1, in the notation of (39)–(40). More precisely, we will check that, for $i = 0, 1, \dots, N$ such that $v_i(\omega) < 1 - \epsilon$,

$$\mathcal{K}_{v_i(\omega)} \geq U_i^e(x_1(\omega), v_1(\omega), \dots, x_i(\omega), v_i(\omega)) - iA \quad (43)$$

with lower game-theoretic probability at least $1 - 2i\epsilon$, where

$$A := 3f(\epsilon) + g(\epsilon)$$

and $g(\epsilon)$ is defined by (38).

We use induction in i . Suppose (43) holds; our goal is to prove (43) with $i+1$ in place of i . We have, for $v_{i+1} < 1 - \epsilon$:

$$\mathcal{K}_{v_{i+1}} \geq \mathcal{K}_{T_{i,J_i}} - f(\epsilon) \quad (44)$$

$$\begin{aligned} &= \mathcal{K}_{T_{i,0}} + \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - \overline{U}_i(X_{i,0}, 0; x_1, v_1, \dots, x_i, v_i) \\ &\quad - f(\epsilon) \\ &\geq \mathcal{K}_{v_i} + \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - U_i^e(x_1, v_1, \dots, x_i, v_i) \\ &\quad - f(\epsilon) - g(\epsilon) \end{aligned} \quad (45)$$

$$\geq \overline{U}_i(X_{i,J_i}, J_i; x_1, v_1, \dots, x_i, v_i) - iA - f(\epsilon) - g(\epsilon) \quad (46)$$

$$\geq \overline{U}_i(X_{i,J_i}, L; x_1, v_1, \dots, x_i, v_i) - iA - 2f(\epsilon) - g(\epsilon) \quad (47)$$

$$= U_i^m(x_1, v_1, \dots, x_i, v_i, X_{i,J_i} \sqrt{S/NL}) - iA - 2f(\epsilon) - g(\epsilon) \quad (48)$$

$$\geq U_i^m(x_1, v_1, \dots, x_i, v_i, x_{i+1}) - iA - 3f(\epsilon) - g(\epsilon) \quad (49)$$

$$\geq U_i^e(x_1, v_1, \dots, x_i, v_i, x_{i+1}, v_{i+1}) - iA - 3f(\epsilon) - g(\epsilon) \quad (50)$$

where:

- the inequality (44) holds for a large enough L and follows from the form (42) of the bets (called off at time $T_{i,L}$) and the uniform continuity of U_i^m (which propagates to \overline{U}_i) with f as modulus of continuity (for all i); the error term $f(\sqrt{S/NL})$ is replaced by the cruder $f(\epsilon)$;
- the inequality (45) follows from the approximate equality (37), whose accuracy is given by (38) (notice that the accuracy (38) is also applicable to (37) with $\lceil \dots \rceil$ in place of $\lfloor \dots \rfloor$); this inequality also relies on the equality $\mathcal{K}_{T_{i,0}} = \mathcal{K}_{v_i}$, which follows from our definition of the bets;
- the inequality (46) holds with lower game-theoretic probability at least $1 - 2i\epsilon$ by the inductive assumption;
- the inequality (47) holds with lower game-theoretic probability at least $1 - \epsilon$ for a large enough L , and follows from Theorem 3.1 of [11] and the uniform continuity of U_i^m with f as modulus of continuity;
- the equality (48) holds by the definition (34);
- the inequality (49) also holds with lower game-theoretic probability at least $1 - \epsilon$ for a large enough L and follows from Theorem 3.1 of [11] and the uniform continuity of U_i^m with f as modulus of continuity.

We can see that the overall chain (44)–(50) holds with lower probability at least $1 - 2(i+1)\epsilon$.

So far we have considered the case $v_{i+1} < 1 - \epsilon$. Now suppose

$$1 - \epsilon \in (v_i(\omega), v_{i+1}(\omega)].$$

As soon as time $1 - \epsilon$ is reached, the strategy stops playing: we will show that with a lower game-theoretic probability arbitrarily close to 1 the goal has been achieved. Indeed, as we saw above,

$$\mathcal{K}_{v_i(\omega)} \gtrsim \overline{U}_i^e(x_1, v_1, \dots, x_i, v_i)$$

with high lower game-theoretic probability. Let us check that

$$\mathcal{K}_{1-\epsilon} \gtrsim F_N(\omega)$$

with high lower game-theoretic probability. This is true since $\mathcal{K}_{1-\epsilon}$ is, with high lower probability, greater than or close to the average of

$$\overline{U}_i^m(x_1, v_1, \dots, x_i, v_i, \xi) \geq \overline{U}_{i+1}^e(x_1, v_1, \dots, x_i, v_i, \xi, 1)$$

$$\begin{aligned}
&= \overline{U}_{i+1}^e(x_1, v_1, \dots, x_i, v_i, \omega(1), 1) \\
&= \overline{U}_N^e(x_1, v_1, \dots, x_i, v_i, \omega(1), 1, \dots, \omega(1), 1) \\
&\geq F_N(\omega) - f(\epsilon)
\end{aligned}$$

(cf. (21) and (24)) over the value ξ at time $(i+1)S/N - \langle \omega \rangle_{1-\epsilon}$ of a Brownian motion started at $\omega(1-\epsilon)$ at time 0 and stopped when it hits level 0, where $\langle \omega \rangle$ is the quadratic variation of ω as defined in [11], Section 8.

To ensure that his capital is always positive, the trader stops playing as soon as his capital hits 0. Increasing his initial capital by a small amount we can make sure that this will never happen (for L sufficiently large). Increasing his initial capital by another small amount we can make sure that he always superhedges F_N and not just with high lower game-theoretic probability. Letting $L \rightarrow \infty$, we obtain $\mathbb{E}^g(F_N) \leq U_0^e$.

4 Conclusion

There is no doubt that this version of the paper makes various unnecessary assumptions. To relax or eliminate those assumptions is a natural direction of further research.

Acknowledgements

The impetus for writing this paper was a series of discussions in April 2015 with Nicolas Perkowski, David Prömel, Martin Huesmann, Alexander M. G. Cox, Pietro Siorpaes, and Beatrice Acciaio during the junior trimester “Optimal transport” held at the Hausdorff Mathematical Centre (Bonn, Germany). They posed the problem of proving or disproving the coincidence of game-theoretic and measure-theoretic probability in the case of the full Wiener space Ω , and this paper gives a positive answer to a radically simplified version of that problem. I am grateful to the organizers of the trimester for inviting me to give a mini-course on game-theoretic probability. Thanks to Gert de Cooman and Jasper de Bock for numerous discussions and their critique of my “narrow” definition of game-theoretic probability (the one given in [11]) as too broad; with apologies to them, this paper experiments with an even broader definition.

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